# Übungsstunde 6:

## Themen:

- Koordinaten in einer bestimmten Basis
- Koordinatentransformation & Basiswechsel
- Abbildungen unter dem Basiswechsel
- Lineare Abbildungen

## Koordinaten in einer bestimmten Basis:

$$\underline{v} = x_1 \cdot \underline{b}^{(1)} + x_2 \cdot \underline{b}^{(2)} + \dots + x_n \cdot \underline{b}^{(n)} \iff \underline{v} \mathrel{\hat=} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_B$$

Beispiel 6.1:

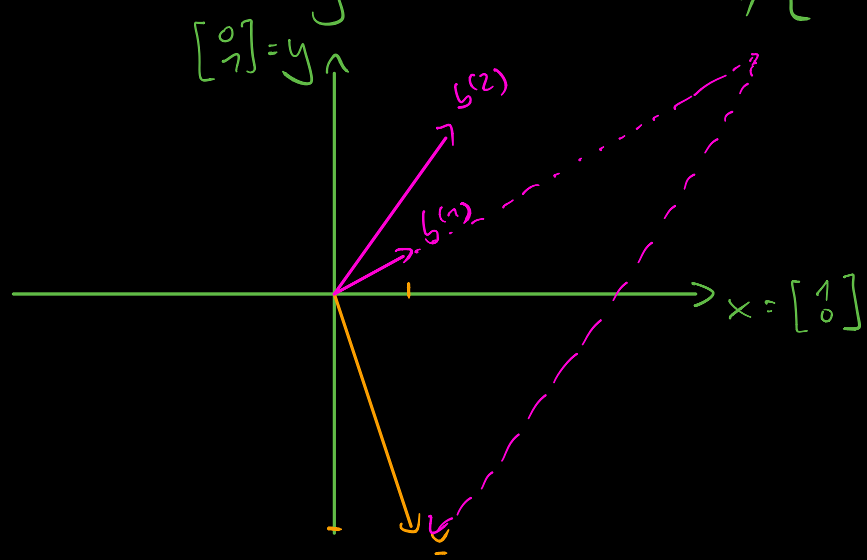
$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x, y \in \mathbb{R} \right\} = \mathbb{R}^2$$

$$B \mathrel{\hat=} \underline{\underline{B}} = \left[ \underline{b}^{(1)} \, \underline{b}^{(2)} \right] = \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}, \quad \underline{v} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}_S$$

Wollen $[\underline{v}]_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, wir wissen $\underline{v} = x_1 \cdot \underline{b}^{(1)} + x_2 \cdot \underline{b}^{(2)}$

$$\Rightarrow \underbrace{\begin{bmatrix} 2 \\ -10 \end{bmatrix}}_{\underline{v}} = x_1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}}_{\underline{\underline{B}}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\underline{x}}$$

$\Rightarrow$ Müssen einfach gaussen $\Rightarrow \underline{x} = \frac{1}{7} \begin{bmatrix} 40 \\ -22 \end{bmatrix}_B = [\underline{v}]_B$



Beispiel 6.2: $U = \{ \underline{x} \in \mathbb{R}^4 \mid x_2 - 2x_3 + x_4 = 0 \}$

$$x_2 - 2x_3 + x_4 = 0 \iff \begin{bmatrix} 0 & \boxed{1} & -2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$x_1 = t \in \mathbb{R}$

$x_3 = s \in \mathbb{R}$

$x_4 = u \in \mathbb{R}$

$x_2 = 2x_3 - x_4$
$\quad = 2s - u$

$$\Rightarrow \mathcal{L} = \left\{ \begin{bmatrix} t \\ 2s-u \\ s \\ u \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; t,s,u \in \mathbb{R} \right\}$$

$$= \left\{ t \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; t,s,u \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Koordinaten $\begin{bmatrix} t \\ s \\ u \end{bmatrix}$

Frage: $B = \{ b^{(1)} = 1, b^{(2)} = x-2, b^{(3)} = x^2 \}$ in $P_2$ (Polynome vom Grad $\leq 2$)

Referenz: $S = \{ e^{(1)} = 1, e^{(2)} = x, e^{(3)} = x^2 \}$

Was sind die Koordinaten des Vektors/Polynoms $p(x) = 7x^2 - 4x + 4$

$$[p(x)]_B = -4(1) - 4(x-2) + 7(x^2) \mathrel{\hat{=}} \begin{bmatrix} -4 \\ -4 \\ 7 \end{bmatrix}_B$$

$$[p(x)]_S = 4 \cdot (1) - 4(x) + 7(x^2) \mathrel{\hat{=}} \begin{bmatrix} 4 \\ -4 \\ 7 \end{bmatrix}_S$$

# Koordinatentransformation & Basiswechsel:

Ausgangslage:

$$[\underline{v}]_{\tilde{B}} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{bmatrix} \xrightarrow[\underline{\underline{T}}]{\text{wollen}} [\underline{v}]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\tilde{B} = \begin{bmatrix} | & | & & | \\ \tilde{b}^{(1)} & \tilde{b}^{(2)} & \cdots & \tilde{b}^{(n)} \\ | & | & & | \end{bmatrix}$$

$$B = \begin{bmatrix} | & | & & | \\ b^{(1)} & b^{(2)} & \cdots & b^{(n)} \\ | & | & & | \end{bmatrix}$$

$$\underline{v} = \tilde{x}_1 \begin{bmatrix} | \\ \tilde{b}^{(1)} \\ | \end{bmatrix} + \tilde{x}_2 \begin{bmatrix} | \\ \tilde{b}^{(2)} \\ | \end{bmatrix} + \dots + \tilde{x}_n \begin{bmatrix} | \\ \tilde{b}^{(n)} \\ | \end{bmatrix}$$

$$= \underline{x_1} \begin{bmatrix} | \\ b^{(1)} \\ | \end{bmatrix} + \underline{x_2} \begin{bmatrix} | \\ b^{(2)} \\ | \end{bmatrix} + \dots + \underline{x_n} \begin{bmatrix} | \\ b^{(n)} \\ | \end{bmatrix}$$

$$= \tilde{x}_1 \underbrace{\left( a_1 \begin{bmatrix} | \\ b^{(1)} \\ | \end{bmatrix} + a_2 \begin{bmatrix} | \\ b^{(2)} \\ | \end{bmatrix} + \dots + a_n \begin{bmatrix} | \\ b^{(n)} \\ | \end{bmatrix} \right)}_{\begin{bmatrix} | \\ \tilde{b}^{(1)} \\ | \end{bmatrix}} + \tilde{x}_2 \underbrace{\left( c_1 \begin{bmatrix} | \\ b^{(1)} \\ | \end{bmatrix} + \dots \right.}_{\begin{bmatrix} | \\ \tilde{b}^{(2)} \\ | \end{bmatrix}}$$

$$= \underbrace{(\tilde{x}_1 a_1 + \tilde{x}_2 c_1 + \tilde{x}_3 d_1 + \dots + \tilde{x}_n z_1)}_{x_1} \begin{bmatrix} | \\ b^{(1)} \\ | \end{bmatrix} + \underbrace{(\tilde{x}_1 a_2 + \dots}_{x_2}$$

$$\Rightarrow \underline{\underline{T}} \, [v]_{\tilde{B}} = [\underline{v}]_B$$

$$\underline{\underline{T}} = \begin{bmatrix} a_1 & c_1 & d_1 & \cdots & z_1 \\ a_2 & c_2 & d_2 & \cdots & z_2 \\ & & \vdots & \ddots & \\ a_n & c_n & d_n & \cdots & z_n \end{bmatrix} = \begin{bmatrix} | & | & \\ [\tilde{b}^{(1)}]_B & [\tilde{b}^{(2)}]_B & \cdots \\ | & | & \end{bmatrix}$$

Bsp. 6.4: Sei $V$ ein Vektorraum mit $\begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix}_B$ $\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}$

$\tilde{B} = \{\tilde{b}^{(1)} = 3t,\ \tilde{b}^{(2)} = 2t^2+2,\ \tilde{b}^{(3)} = t+1\}$ die alte Basis

$B = \{b^{(1)} = 1,\ b^{(2)} = t,\ b^{(3)} = t^2\}$ die neue Basis

Müssen nur die Koordinatendarstellung der alten Basis in der neuen Basis finden $\Rightarrow$ Spalten der Basiswechselmatrix $T$:

$$\left.\begin{aligned} \tilde{b}^{(1)} &= 3t = 0\cdot(1) + 3(t) + 0(t^2) \mathrel{\hat=} \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}_B \\ \tilde{b}^{(2)} &= 2t^2+2 = 2\cdot(1) + 0(t) + 2(t^2) \mathrel{\hat=} \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}_B \\ \tilde{b}^{(3)} &= t+1 = 1\cdot(1) + 1(t) + 0(t^2) \mathrel{\hat=} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_B \end{aligned}\right\} \underline{\underline{T}} = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$$

$$\Rightarrow \underline{\underline{T}}\,[\underline{v}]_{\tilde{B}} = [\underline{v}]_B$$

$$\underline{\underline{S}}\,[\underline{v}]_B = [\underline{v}]_{\tilde{B}}$$

$$\underline{\underline{S}} = \underline{\underline{T}}^{-1}$$

Alternative Schreibweise

$\{\underline{\underline{T}}_{B\tilde{B}} \cdot {}_{\tilde{B}}\underline{v} = {}_B\underline{v}\}$

Formeln:

"Koordinaten in der alten Basis zur neuen Basis": $\underline{\underline{T}}$

$$\underline{\underline{T}} \cdot [\underline{v}]_{\tilde{B}} = [\underline{v}]_{B} \, , \quad \underline{\underline{B}} \cdot \underline{\underline{T}} = \underline{\underline{\tilde{B}}}$$

"Neue Basis" · "BWM alt zu neu" = "Alte Basis"

↳ unintuitiv!

"Koordinaten in der neuen Basis zur alten Basis":

$$\underline{\underline{S}} \cdot [\underline{v}]_{B} = [\underline{v}]_{\tilde{B}} \, , \quad \underline{\underline{\tilde{B}}} \cdot \underline{\underline{S}} = \underline{\underline{B}}$$

Beweisskizze:

$$\underline{\underline{\tilde{B}}} = \underline{\underline{B}} \cdot \underline{\underline{T}} = \begin{bmatrix} | & | & | \\ b^{(1)} & b^{(2)} & b^{(3)} \\ | & | & | \end{bmatrix} \cdot \begin{bmatrix} \boxed{\begin{matrix} t_{11} \\ t_{21} \\ t_{31} \end{matrix}} & \begin{matrix} t_{12} \\ t_{22} \\ t_{32} \end{matrix} & \begin{matrix} t_{13} \\ t_{23} \\ t_{33} \end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} | & | & | \\ b^{(1)} & b^{(2)} & b^{(3)} \\ | & | & | \end{bmatrix} \cdot \begin{bmatrix} \boxed{\begin{matrix} | \\ [\tilde{b}^{(1)}]_B \\ | \end{matrix}} & \begin{matrix} | \\ [\tilde{b}^{(2)}]_B \\ | \end{matrix} & \begin{matrix} | \\ [\tilde{b}^{(3)}]_B \\ | \end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} \underbrace{t_{11} \begin{bmatrix} | \\ b^{(1)} \\ | \end{bmatrix} + t_{21} \begin{bmatrix} | \\ b^{(2)} \\ | \end{bmatrix} + t_{31} \begin{bmatrix} | \\ b^{(3)} \\ | \end{bmatrix}}_{[\tilde{b}^{(1)}]} & \underbrace{t_{12} \begin{bmatrix} | \\ b^{(1)} \\ | \end{bmatrix} + \dots}_{[\tilde{b}^{(2)}]} \end{bmatrix}$$

$$\underline{\underline{B}} \cdot \underline{\underline{T}} = \underline{\underline{\tilde{B}}} \iff \underline{\underline{B}} \,|\, \underline{\underline{\tilde{B}}} \overset{\text{Gauss}}{\underset{\text{Jordan}}{\Longrightarrow}} \underline{\underline{I}} \,|\, \underline{\underline{T}}$$

$$\underline{\underline{T}} = \underline{\underline{B}}^{-1} \underline{\underline{\tilde{B}}} \qquad \underline{\underline{I}} \cdot \underline{\underline{T}} = \underline{\underline{T}}$$

## Abbildungen unter der BWM:

$\mathcal{F}: V \to V$ linear

wichtig

- $[\mathcal{F}]_{\tilde{B}} = \underline{\underline{\tilde{A}}}$ Darstellungsmatrix von $\mathcal{F}$ in der alten Basis $\tilde{B}$

- $[\mathcal{F}]_{B} = \underline{\underline{A}}$ "

  " " " "

  neuen " $B$

- $\underline{\underline{T}}$ Basiswechselmatrix $\tilde{B} \to B$

## Kommutatives Diagram:

$$
\begin{tikzcd}
\underline{\tilde{x}} & & \underline{\tilde{y}} = \underline{\underline{\tilde{A}}}\,\underline{\tilde{x}} \\
\tilde{B} \arrow[rr, "{[\mathcal{F}]_{\tilde{B}} = \underline{\underline{\tilde{A}}}}"] \arrow[d, "\underline{\underline{T}}"] & & \tilde{B} \\
B \arrow[rr, "{[\mathcal{F}]_{B} = \underline{\underline{A}}}"'] & & B \arrow[u, "\underline{\underline{S}} = \underline{\underline{T}}^{-1}"'] \\
\underline{x} & & \underline{y} = \underline{\underline{A}}\,\underline{x}
\end{tikzcd}
$$

$$\underline{\underline{[\mathcal{F}]_{\tilde{B}}}} = \underline{\underline{S}}\,\underline{\underline{[\mathcal{F}]_{B}}}\,\underline{\underline{T}}$$

$$\neq T\,[\mathcal{F}]_{B}\,S$$